

# ON J. C. C. NITSCHÉ'S TYPE INEQUALITY FOR HYPERBOLIC SPACE $\mathbf{H}^3$

DAVID KALAJ

ABSTRACT. Let  $\mathbf{H}^3$  be the hyperbolic space identified with the unit ball  $\mathbf{B}^3 = \{x \in \mathbf{R}^3 : |x| < 1\}$  with the Poincaré metric  $d_h$  and assume that  $\mathcal{A}(x_0, p, q) := \{x : p < d_h(x, x_0) < q\} \subset \mathbf{H}^3$  is an hyperbolic annulus with the inner and outer radii  $0 < p < q < \infty$ . We prove that if there exists a proper hyperbolic harmonic mapping between annuli  $\mathcal{A}(x_0, a, b)$  and  $\mathcal{A}(y_0, \alpha, \beta)$  in the hyperbolic space  $\mathbf{H}^3$ , then  $\beta/\alpha > 1 + \psi(a, b)$ , where  $\psi$  is a positive function.

## 1. INTRODUCTION

**1.1. Background and statement of the main result.** In this paper by  $\mathbf{A}(a, b)$  we denote the annulus  $\{x \in \mathbf{R}^n : a < |x| < b\}$  in the euclidean space  $\mathbf{R}^n$ ,  $n \geq 2$ . The unit ball is defined by  $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$  and the unit sphere is defined by  $S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$  (here  $x = (x_1, \dots, x_n)$  and  $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ ). Fifty years ago J. C. C. Nitsche [14], studying the minimal surfaces and inspired by radial harmonic mappings between annuli asked a question whether the existence of a euclidean harmonic mapping between circular annuli  $\mathbf{A}(a, 1)$  and  $\mathbf{A}(\alpha, 1)$  in  $\mathbf{R}^2$  is equivalent with the simple inequality

$$(1.1) \quad \alpha \leq \frac{2a}{1+a^2}.$$

This question is answered recently in positive by Iwaniec, Kovalev and Onninen in [5]. The Nitsche conjecture is deeply rooted in the theory of doubly connected minimal surfaces. Some partial results have been obtained previously by Lyzzaik [13], Weitsman [17] and the author [7]. On the other hand in [3] and in [9] is treated the same problem for the harmonic mappings w.r.t hyperbolic and Riemann metric in two-dimensional hyperbolic space and in two-dimensional Riemann sphere respectively. In [9, 6] the author treated the three-dimensional case and obtained an inequality for euclidean harmonic mappings between annuli on  $\mathbf{R}^3$ . The  $n$ -dimensional generalization of conjectured inequality (1.1) is

$$\alpha \leq \frac{2a}{n-1+a^n}$$

and is inspired by radial harmonic mappings

$$(1.2) \quad f(x) = \left( \frac{1-a^{n-1}\alpha}{1-a^n} + \frac{a^{n-1}\alpha-a^n}{(1-a^n)|x|^n} \right) x$$

between annuli  $\mathbf{A}(a, 1)$  and  $\mathbf{A}(\alpha, 1)$  (c.f. [6]). The last conjectured inequality for  $n \geq 3$  remains an open problem.

---

2000 *Mathematics Subject Classification.* Primary 58E20.

*Key words and phrases.* Harmonic mappings, Spherical annuli, Hyperbolic geometry.

The existence of harmonic mappings between certain annuli in the two dimensional euclidean space is deeply related to the existence of minimizers of Dirichlet integral without boundary data for differentiable mappings between annuli. Further it is shown in [2] that the minimizer of Dirichlet integral w.r.t. euclidean metric of certain deformations between annuli  $\mathbf{A}(a, 1)$  and  $\mathbf{A}(\alpha, 1)$  is a homeomorphism if and only if the inequality (1.1) is satisfied. In this case the minimizer is the harmonic diffeomorphism given by (1.2) ( $n = 2$ ). See also [8, 10] for some generalization of the previous problem to radial metrics. In multidimensional setting (when  $n \geq 3$ ), the minimization problem of Dirichlet energy without boundary data is essentially different from the case  $n = 2$ . It seems that in this case the minimization of  $n$ -energy is more appropriate. Then the appropriate Euler-Lagrange equations reads as a  $n$ -harmonic equation. In [4] Iwaniec and Onninen formulated a J. C. C. Nitsche type inequality for  $n$ -harmonic mappings and shown that under these inequality the absolute minimizers of Dirichlet energy are radial  $n$ -harmonic mappings. One of advantages of  $n$ -harmonic mappings for  $n \geq 3$  is that they are invariant under Möbius transformations of the space. This property the class of  $n$ -harmonic mappings share with hyperbolic harmonic mappings. Moreover hyperbolic harmonic mappings are invariant under Möbius transformations of the domain as well as of the image domain.

In this paper we consider hyperbolic harmonic mappings between certain subsets of the hyperbolic space  $\mathbf{H}^3$ . Li and Tam in [11, 12] established the existence and regularity of proper hyperbolic harmonic mappings between  $\mathbf{H}^n$  onto  $\mathbf{H}^m$ , satisfying certain conditions in the ideal boundary  $\mathbf{S}^{n-1}$ .

The purpose of this paper is to study J. C. C. Nitsche type problem for the harmonic mappings between domains of the hyperbolic spaces  $\mathbf{H}^3$ .

In this paper we prove the following theorem:

**Theorem 1.1.** *Let  $\mathbf{A} = \mathbf{A}(a, b)$ ,  $\mathbf{A}' = \mathbf{A}(\alpha, \beta) \subset \mathbf{B}^3$ ,  $0 < a < b < 1$ ,  $0 < \alpha < \beta < 1$  be spherical annuli endowed with the hyperbolic metric of the unit ball. If there exists a proper hyperbolic harmonic mapping  $u$  of  $\mathbf{A}$  onto  $\mathbf{A}'$  then*

$$(1.3) \quad \frac{(1 - \alpha^2)^2}{4\alpha(1 + \alpha^2)} \log \left[ \frac{1 + \beta}{1 + \alpha} \frac{1 - \alpha}{1 - \beta} \right] \geq \left( -1 + \frac{a}{b} + \log \frac{b}{a} \right) : \left( 1 + \log \frac{1 - a^2}{1 - b^2} \right).$$

Recall that a mapping  $f : X \rightarrow Y$  between two topological spaces is proper if and only if the preimage of every compact set in  $Y$  is compact in  $X$ .

**1.2. Hyperbolic harmonic mappings.** In general, if the metrics of two non-compact manifolds  $M^m$  and  $N^n$  are given locally by

$$ds_M^2 = \sum_{i,j}^m g_{ij} dx^i dx^j$$

and

$$ds_N^2 = \sum_{\alpha,\beta=1}^n h_{\alpha\beta} du^\alpha du^\beta$$

respectively, then the energy-density function of a  $C^1$  map  $u : M \rightarrow N$  is defined by

$$e(u)(x) = \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^n g^{ij}(x) h_{\alpha\beta}(u(x)) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j},$$

and the total energy of  $u$  is given by

$$E(u) = \int_M e(u)(x) dx.$$

The harmonic-map equation from  $M$  into  $N$ , which is the Euler-Lagrange equation for critical points of the total energy functional, can be written as

$$\Delta u^\alpha(x) = - \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^n \Gamma_{\beta\gamma}^\alpha(u(x)) g^{ij}(x) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j},$$

for all  $1 \leq \alpha \leq m$ , where the  $\Gamma_{\beta\gamma}^\alpha$  are the Christoffel symbols of  $N$ . Here  $\Delta$  is Laplace-Beltrami operator on  $M^m$ . We refer to the monograph [15] for some important properties of harmonic mappings.

The hyperbolic space  $\mathbf{H}^n$  is identified with  $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$  with the Poincaré metric tensor given by

$$ds_{\mathbf{H}}^2(x) = \frac{4|dx|^2}{(1-|x|^2)^2}$$

which in polar coordinates can be written as

$$ds_{\mathbf{H}}^2(x) = 4 \frac{d\rho^2 + \rho^2 \sum_{i,j=2}^n g_{ij} d\eta^i d\eta^j}{(1-\rho^2)^2}$$

where  $\rho^2 = |x|^2 = \sum_{i=1}^n x_i^2$  and  $\sum_{i,j=2}^n g_{ij} d\eta^i d\eta^j$  is the standard metric tensor on the unit sphere  $S^{n-1}$ .

Polar coordinates are natural to use when working with annuli. We associate with any point  $x \in \mathbf{B}_o^n$  a pair of polar coordinates

$$(1.4) \quad (r, \omega) \in (0, 1) \times \mathbf{S}^{n-1} \sim \mathbf{B}_o^n$$

where  $r = |x|$  is referred to as the radial distance and  $\omega = \frac{x}{|x|}$  as the spherical coordinate of  $x$ . Obviously  $x = r\omega$  and the volume element in polar coordinates reads as  $dV(x) = r^{n-1} dr d\mathcal{H}^{n-1}(\omega)$ , where  $\mathcal{H}^{n-1}$  is the  $n-1$ -dimensional Hausdorff surface measure.

If the polar coordinates of a point  $x \in \mathbf{B}^n$  are  $(r, \omega)$ , then the geodesic polar coordinates are  $(2 \tanh^{-1}(|x|), \omega)$  and they will be of crucial importance for our approach. In [11] Li and Tam computed the coefficients of tension field of a mapping

$$u(x) = r(x)\Theta(x) = r(x)(\theta^1(x), \dots, \theta^n(x)) : \mathbf{X} \rightarrow \mathbf{B}^n, \quad \mathbf{X} \subset \mathbf{B}^m$$

in polar coordinates as follows

$$(1.5) \quad \begin{aligned} \tau(u)^1 &= \frac{(1-\rho^2)^2}{4} \Delta_0 r \\ &+ \frac{1}{4} \left( 2(m-2)(1-\rho^2) \rho \frac{\partial \rho}{\partial \rho} \right. \\ &\quad \left. + \frac{r(1-\rho^2)^2 (2|\nabla_0 r|^2 - (1+r^2) \sum_{p,q=2}^n h_{pq} \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle)}{1-r^2} \right) \end{aligned}$$

and

$$(1.6) \quad \begin{aligned} \tau(u)^s = & \frac{(1-\rho^2)^2}{2} \left( \Delta_0 \theta^s + \sum_{p,q=2}^n \tilde{\Gamma}_{pq}^s \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle \right) \\ & + \frac{1-\rho^2}{2} \left( (m-2)\rho \frac{\partial \theta^s}{\partial \rho} + \frac{(1+r^2)(1-\rho^2) \langle \nabla_0 r, \nabla_0 \theta^s \rangle}{r(1-r^2)} \right) \end{aligned}$$

for  $s \geq 2$ . Here  $\Delta_0$  denotes the euclidean Laplacian and  $\nabla_0$  denotes the euclidean gradient and the  $\tilde{\Gamma}_{pq}^s$  denote the Christoffel symbols with respect to the standard metric tensor of  $S^{n-1}$ . The mapping  $u$  is harmonic if  $\tau(u)^s = 0$  for  $1 \leq s \leq n$ . We will use only the relation (1.5). This in particular mean that, the results of this paper can be formulated in slightly more general setting.

The set of isometries of the hyperbolic space is a Kleinian group subgroup of all Möbius transformations of the extended space  $\bar{\mathbf{R}}^n$  onto itself denoted by  $\mathbf{Conf}(\mathbf{B}^n) = \mathbf{Isom}(\mathbf{H}^n)$ . We refer to the Ahlfors' book [1] for detailed survey to this class of important mappings. We recall that the hyperbolic metric  $d_h(z, w)$  of the unit ball  $\mathbf{B}^n$  is defined by

$$\tanh \frac{d_h(x, y)}{2} = \frac{|x - y|}{[x, y]},$$

where  $[x, y]^2 := 1 + |x|^2|y|^2 - 2 \langle x, y \rangle$ . In particular  $d_h(0, x) = 2 \tanh^{-1}(|x|)$ . Since the harmonicity is an isometric invariant, we obtain the following proposition.

**Proposition 1.2.** *If  $u : \mathbf{X} \rightarrow \mathbf{Y}$  is a harmonic mapping between the domains  $\mathbf{X}$  and  $\mathbf{Y}$  of the unit ball and  $f, g \in \mathbf{Conf}(\mathbf{B}^n)$ , then  $f \circ u \circ g$  is a harmonic mapping between  $g(\mathbf{X})$  and  $f(\mathbf{Y})$ .*

Let  $x_0 \in \mathbf{H}^3$  and assume that  $0 < p < q < \infty$ . Then the set  $\mathcal{A}(x_0, p, q) := \{x \in \mathbf{H}^3 : p < d_h(x, x_0) < q\}$  is called the hyperbolic annulus. Moreover  $\mathcal{A}(0, p, q) = \mathbf{A}(p', q')$ ,  $p = \tanh \frac{p'}{2}$  and  $q = \tanh \frac{q'}{2}$ . Having in mind the previous observation, together with Proposition 1.2 we obtain the following reformulation of Theorem 1.1

**Theorem 1.3.** *Let  $\mathcal{A} = \mathcal{A}(x_0, a', b')$ ,  $\mathcal{A}' = \mathcal{A}(y_0, \alpha', \beta') \subset \mathbf{B}^3$ ,  $0 < a' < b' < \infty$ ,  $0 < \alpha' < \beta' < \infty$  be spherical annuli endowed by hyperbolic metric of the unit ball. If there exists a proper hyperbolic harmonic mapping  $u$  of  $\mathcal{A}$  onto  $\mathcal{A}'$  then*

$$(1.7) \quad \frac{\beta'}{\alpha'} \geq 1 + \frac{\sinh(2\alpha')}{\alpha'} \frac{\log[\coth \frac{a'}{2} \tanh \frac{b'}{2}] + \coth \frac{b'}{2} \tanh \frac{a'}{2} - 1}{1 + 2 \log[\cosh \frac{b'}{2} \operatorname{sech} \frac{a'}{2}]}$$

*Remark 1.4.* Since  $\frac{\sinh(2\alpha')}{\alpha'} > 2$ , from (1.7), we obtain the following

$$\frac{\alpha'}{\beta'} < \varphi(a', b') < 1, \text{ (c.f. (1.1), for } n = 2 \text{ and euclidean metric).}$$

## 2. PRELIMINARY RESULTS

For a matrix  $A = \{a_{ij}\}_{i,j=1}^n$  we define the Hilbert-Schmidt and geometric norm as follows

$$\|A\|_2 = \sqrt{\sum_{i,j} a_{ij}^2} \text{ and } \|A\| = \sup\{|Ax| : |x| = 1\},$$

respectively. Let  $\lambda_1^2 \leq \dots \leq \lambda_n^2$  be the eigenvalues of the matrix  $A^T A$ . Then

$$\|A\|_2 = \sqrt{\sum_{i=1}^n \lambda_i^2} \text{ and } \|A\| = \lambda_n$$

and

$$\det A = \prod_{k=1}^n \lambda_k.$$

We say that  $A$  is  $K$ -quasiconformal, where  $K \geq 1$ , if  $\lambda_n \leq K\lambda_1$ .

**Lemma 2.1.** *There hold the sharp inequality*

$$(2.1) \quad |Ax_1 \times \dots \times Ax_{n-1}| \leq \left[ \frac{K^2}{1 + (n-1)K^2} \right]^{(n-1)/2} \|A\|_2^{n-1} |x_1 \times \dots \times x_{n-1}|,$$

where  $1 \leq K \leq \infty$  is the constant of quasiconformality of  $A$ . If  $A$  is an orthogonal transformation, then in (2.1) we have equality with  $K = 1$ . If  $A$  is singular, then  $K = \infty$  and we make use of the convention  $\frac{K^2}{1+(n-1)K^2} = \frac{1}{n-1}$ .

*Proof.* Assume firstly that  $A$  is a nonsingular matrix. Then  $A$  is  $K$ -quasiconformal for some  $K < \infty$ . Let  $x_i = \sum_{j=1}^n x_{ij} e_j$ ,  $i = 1, \dots, n-1$ . Then

$$Ax_1 \times \dots \times Ax_{n-1} = \sum_{\sigma} \varepsilon_{\sigma} x_{1,\sigma_1} \dots x_{n-1,\sigma_{n-1}} A e_1 \times \dots \times A e_{n-1}.$$

It follows that

$$(2.2) \quad Ax_1 \times \dots \times Ax_{n-1} = \tilde{A}(x_1 \times \dots \times x_{n-1}).$$

Here  $\tilde{A}$  is the adjugate of  $A$ , which for nonsingular matrix  $A$  satisfies the relation  $\tilde{A} = \det A \cdot A^{-1}$ . As  $A$  is  $K$  quasiconformal,  $\tilde{A}$  is  $K$  quasiconformal as well. Let  $\lambda_1^2 \leq \dots \leq \lambda_n^2$  be the eigenvalues of the matrix  $A^T A$ .  $A$  is  $K$ -quasiconformal if and only if

$$(2.3) \quad \frac{\lambda_n}{\lambda_1} \leq K.$$

From  $\tilde{A} = \det A \cdot A^{-1}$ , it follows that

$$\tilde{\lambda}_k = \det A \cdot \frac{1}{\lambda_k}, \text{ and } \tilde{\lambda}_n \leq \tilde{\lambda}_{n-1} \leq \dots \leq \tilde{\lambda}_1$$

and consequently

$$\frac{\tilde{\lambda}_1}{\tilde{\lambda}_n} \leq K.$$

From (2.2) we obtain

$$(2.4) \quad |Ax_1 \times \dots \times Ax_{n-1}| \leq \|\tilde{A}\| \cdot |x_1 \times \dots \times x_{n-1}|.$$

Furthermore

$$(2.5) \quad \|\tilde{A}\| = \tilde{\lambda}_1 = \frac{\det A}{\lambda_1} = \prod_{k=2}^n \lambda_k.$$

On the other hand

$$(2.6) \quad \|A\|_2 = \sqrt{\sum_{k=1}^n \lambda_k^2}.$$

From G-A inequality we have

$$(2.7) \quad \frac{\prod_{k=2}^n \lambda_k}{\left(\sqrt{\sum_{k=1}^n \lambda_k^2}\right)^{n-1}} \leq \frac{1}{(n-1)^{(n-1)/2}} \frac{\left(\sqrt{\sum_{k=2}^n \lambda_k^2}\right)^{n-1}}{\left(\sqrt{\sum_{k=1}^n \lambda_k^2}\right)^{n-1}} \\ = \left(\frac{B}{(n-1)(B + \lambda_1^2)}\right)^{(n-1)/2},$$

where  $B = \sum_{k=2}^n \lambda_k^2$ . Since  $\lambda_k$  is an increasing sequence, from (2.3) we have

$$(2.8) \quad \lambda_1^2 \geq \lambda_k^2 / K^2, \quad k = 2, \dots, n.$$

Summing the inequalities (2.8) we obtain

$$(2.9) \quad \lambda_1^2 \geq \frac{B}{(n-1)K^2}.$$

From (2.7), (2.9), (2.5) and (2.6) we obtain

$$\frac{\|\tilde{A}\|}{\|A\|_2^{n-1}} \leq \left[ \frac{K^2}{1 + (n-1)K^2} \right]^{(n-1)/2}.$$

This in view of (2.4), completes the proof of inequality of lemma. To show the sharpness of the inequality, take  $A(x) = (x_1, Kx_2, \dots, Kx_n)$ . Then  $A$  is  $K$ -quasiconformal. Moreover

$$|Ae_2 \times \dots \times Ae_n| = K^{n-1}$$

and

$$\left[ \frac{K^2}{1 + (n-1)K^2} \right]^{(n-1)/2} \|A\|_2^{n-1} |e_2 \times \dots \times e_n| = K^{n-1}.$$

Since the set of singular matrices is nowhere dense and closed subset of  $M_{n \times n}$ , for a singular matrix  $A$  there exists a sequence of positive real numbers  $\epsilon_k$  converging to zero such that  $A_k = A + \epsilon_k I$  is a nonsingular matrix, where  $I$  is the identity matrix. Moreover the constants of quasiconformality  $K_k$  of  $A_k$  tend to  $\infty$ . By applying the previous proof to  $A_k$  we obtain the inequality (2.1) for  $K = \infty$ . The inequality (2.1) is attained for  $A(x) = (0, x_2, \dots, x_n)$ .  $\square$

**Proposition 2.2.** *Let  $u$  be a  $C^1$  surjection between the spherical rings  $\mathbf{A}(a, b)$  and  $\mathbf{A}(\alpha, \beta)$ , and let  $\Theta = (\theta^1, \dots, \theta^n) = u/|u|$ . Let  $P^{n-1}$  be a closed  $n-1$  dimensional hyper-surface that separates the components of the set  $\mathbf{A}^C(a, b)$ . Then*

$$(2.10) \quad \int_{P^{n-1}} \|D\Theta\|_2^{n-1} d\mathcal{H}^{n-1} \geq (n-1)^{\frac{n-1}{2}} \omega_{n-1},$$

and

$$(2.11) \quad \int_{\mathbf{A}(a,b)} \|D\Theta\|_2^{n-1} dV \geq (n-1)^{\frac{n-1}{2}} (b-a) \omega_{n-1},$$

where  $\omega_{n-1}$  denote the measure of  $S^{n-1}$  and  $D\Theta = \{\theta_{x_i}^j\}_{i,j=1}^n$  is the differential matrix of  $\Theta$ . Moreover  $d\mathcal{H}^{n-1}$  is the  $n-1$ -dimensional Hausdorff surface measure and  $dV$  is the volume element.

*Proof.* Let  $K^{n-1}$  be an  $n-1$ -dimensional rectangle and let  $g : K^{n-1} \rightarrow P^{n-1}$  be a parametrization of  $P^{n-1}$ . Then the function  $\Theta \circ g$  is a differentiable surjection from  $K^{n-1}$  onto the unit sphere  $S^{n-1}$ . Then we have

$$\int_{K^{n-1}} D_{\Theta \circ g} dV \geq \omega_{n-1}.$$

(cf. [7, p. 245]). According to Lemma 2.1 (for  $K = \infty$ ), we obtain

$$\begin{aligned} D_{\Theta \circ g}(x) &= \left| D\Theta(g(x)) \frac{\partial g(x)}{\partial x_1} \times \cdots \times D\Theta(g(x)) \frac{\partial g(x)}{\partial x_{n-1}} \right| \\ &\leq (n-1)^{\frac{1-n}{2}} \|D\Theta(g(x))\|_2^{n-1} D_g(x). \end{aligned}$$

Hence we obtain

$$(n-1)^{\frac{n-1}{2}} \omega_{n-1} \leq \int_{K^{n-1}} \|D\Theta(g(x))\|_2^{n-1} D_g(x) dV(x) = \int_{P^{n-1}} \|D\Theta(\zeta)\|_2^{n-1} d\mathcal{H}^{n-1}(\zeta).$$

Thus we have proved (2.10). It follows that

$$\int_{\mathbf{A}(a,b)} \|D\Theta\|_2^{n-1} dV = \int_a^b \left( \int_{S^{n-1}(0,t)} \|D\Theta\|_2^{n-1} d\mathcal{H}^{n-1} \right) dt \geq (n-1)^{\frac{n-1}{2}} (b-a) \omega_{n-1}.$$

The proof of the proposition has been completed.  $\square$

### 3. THE PROOF OF MAIN RESULT

One of key formulas follows from following lemma

**Lemma 3.1.** *Let  $u(x) = r(x)\Theta(x) : \mathbf{A} \rightarrow \mathbf{A}'$  be a hyperbolic harmonic mapping between the domains  $\mathbf{A}$  and  $\mathbf{A}'$  of the hyperbolic space  $\mathbf{B}^n$  and assume that  $R(x) = 2 \tanh^{-1}(r(x))$  and  $\rho = |x|$ . Then*

$$(3.1) \quad \Delta_0 R + \frac{2(n-2)\rho}{(1-\rho^2)} \frac{\partial R}{\partial \rho} = \frac{\sinh(2R)}{2} \|D\Theta\|_2^2,$$

where  $\Delta_0$  and  $\|D\Theta\|_2$  are euclidean Laplacian and Hilbert-Schmidt norm of differential matrix respectively.

*Proof.* Since  $u$  is harmonic, from the equation  $\tau(u)^1 = 0$ , where  $\tau(u)^1$  is defined in (1.5), we obtain

$$\Delta_0 r + \left( 2(n-2)(1-\rho^2)^{-1} \rho \frac{\partial \rho}{\partial \rho} + \frac{r(2|\nabla_0 r|^2 - (1+r^2) \sum_{p,q=2}^n h_{pq} \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle)}{1-r^2} \right) = 0.$$

Let  $g(q) = \tanh(q/2)$ . Then

$$\Delta_0 r = g''(R(x)) |\nabla_0 R|^2 + g'(R(x)) \Delta_0 R$$

and

$$|\nabla_0 r|^2 = (g')^2 |\nabla_0 R|^2.$$

Since

$$\frac{2g}{1-g^2} = \sinh(R) = -\frac{g''}{g'^2}$$

it follows that

$$\frac{1}{2} \operatorname{sech}^2 \frac{R}{2} \Delta_0 R + (n-2) \operatorname{sech}^2 \frac{R}{2} \frac{\rho}{1-\rho^2} \frac{\partial R}{\partial \rho} = \frac{r((1+r^2) \sum_{p,q=2}^n h_{pq} \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle)}{1-r^2},$$

i.e.

$$\Delta_0 R + 2(n-2)(1-\rho^2)^{-1}\rho \frac{\partial R}{\partial \rho} = \frac{\sinh(2R)}{2} \sum_{p,q=2}^n h_{pq} \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle,$$

which can be written as

$$\Delta_0 R + 2(n-2)(1-\rho^2)^{-1}\rho \frac{\partial R}{\partial \rho} = \frac{\sinh(2R)}{2} \|D\Theta\|_2^2.$$

□

*Proof of Theorem 1.3 (Theorem 1.1).* Let  $\alpha' = 2 \tanh^{-1} \alpha$  and  $\beta' = 2 \tanh^{-1} \beta$ . Let  $\varphi_k : [\alpha', \beta'] \mapsto [\alpha', \beta']$  be a sequence of non decreasing functions, constant in some small neighborhood of  $\alpha'$ , for example in  $[\alpha', \alpha' + (\beta' - \alpha')/k]$  and satisfying the following conditions

$$(3.2) \quad 0 \leq \varphi'_k(R) \rightarrow 1 \text{ and } 0 \leq \varphi''_k(R) \rightarrow 0 \text{ as } k \rightarrow \infty$$

for every  $R \in [\alpha', \beta']$ . (See [7] for an example of such sequence). Let  $R_k$  be a function defined on  $\{x : a < |x| < b\}$  by  $R_k(x) = \varphi_k(R(x))$ . Then

$$(3.3) \quad \Delta_0 R_k(x) = \varphi''_k(R(x)) |\nabla R(x)|^2 + \varphi'_k(R(x)) \Delta_0 R(x).$$

Therefore

$$\begin{aligned} \Delta_0 R_k + 2(n-2)(1-\rho^2)^{-1}\rho \frac{\partial R_k}{\partial \rho} &= \varphi''_k(R(x)) |\nabla R(x)|^2 + \varphi'_k(R(x)) \Delta_0 R(x) \\ &\quad + \varphi'_k(R(x)) 2(n-2)(1-\rho^2)^{-1}\rho \frac{\partial R}{\partial \rho} \\ &= \varphi''_k(R(x)) |\nabla R(x)|^2 + \varphi'_k(R(x)) \frac{\sinh(2R_k)}{2} \|D\Theta_k\|_2^2. \end{aligned}$$

Thus

$$(3.4) \quad \Delta_0 R_k + 2(n-2)(1-\rho^2)^{-1}\rho \frac{\partial R_k}{\partial \rho} \geq 0$$

for every  $k$ . By (3.3) and (3.2) it follows at once that

$$\Delta_0 R_k(x) \rightarrow \Delta_0 R(x) \text{ as } k \rightarrow \infty$$

for every  $x \in \mathbf{A}(a, b)$ . Similarly we obtain

$$\frac{\partial R_k}{\partial \rho}(x) \rightarrow \frac{\partial R}{\partial \rho}(x) \text{ as } k \rightarrow \infty$$

uniformly on  $\{x : |\zeta| = s\}$  for every  $s \in (a, b)$ . By applying Green's formula for  $R_k$  on  $\{x : a \leq |x| \leq s\}$ , we obtain

$$\int_{|\zeta|=s} \frac{\partial R_k}{\partial \rho} d\mathcal{H}^{n-1}(\zeta) - \int_{|\zeta|=a} \frac{\partial R_k}{\partial \rho} d\mathcal{H}^{n-1}(\zeta) = \int_{a \leq |x| \leq s} \Delta_0 R_k dV(x).$$

Since the function  $R_k$  is constant in some neighborhood of the sphere  $|\zeta| = a$ , it follows that for  $a < s < b$  and large enough  $k$

$$\int_{|\zeta|=s} \frac{\partial R_k}{\partial \rho} d\mathcal{H}^{n-1}(\zeta) = \int_{a \leq |x| \leq s} \Delta_0 R_k dV(x).$$



Therefore

$$\begin{aligned} \int_{|\zeta|=s} \frac{\partial R_k}{\partial \rho} d\mathcal{H}^{n-1}(\zeta) + \int_{A_s} 2(n-2)(1-\rho^2)^{-1} \rho \frac{\partial R_k}{\partial \rho} dV(x) \\ = \int_{\alpha \leq |x| \leq s} \left[ \Delta_0 R_k + 2(n-2)(1-\rho^2)^{-1} \rho \frac{\partial R_k}{\partial \rho} \right] dV(x). \end{aligned}$$

Further for  $\omega \in S^{n-1}$

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_a^s \frac{\rho^n}{1-\rho^2} \frac{\partial R_k}{\partial \rho}(s\omega) d\rho \\ = \lim_{k \rightarrow \infty} \left[ \frac{s^n R_k(s\omega)}{1-s^2} - \frac{a^n R_k(a\omega)}{1-a^2} - \int_a^s \frac{\rho^{n-1}(n+2\rho^2-n\rho^2)}{(1-\rho^2)^2} R_k(\rho\omega) d\rho \right] \\ \leq \frac{s^n}{1-s^2} \beta' - \frac{a^n}{1-a^2} \alpha' - \int_a^s \frac{\rho^{n-1}(n+2\rho^2-n\rho^2)}{(1-\rho^2)^2} \alpha' d\rho \\ = \frac{s^n}{1-s^2} (\beta' - \alpha'). \end{aligned}$$

Therefore

$$\begin{aligned} (3.5) \quad \int_{|\zeta|=s} \frac{\partial R}{\partial \rho} d\mathcal{H}^{n-1}(\zeta) + 2(n-2) \int_{S^{n-1}} \frac{s^n}{1-s^2} (\beta' - \alpha') d\mathcal{H}^{n-1} \\ \geq \limsup_{k \rightarrow \infty} \int_{\alpha \leq |x| \leq s} \left[ \Delta_0 R_k + 2(n-2)(1-\rho^2)^{-1} \rho \frac{\partial R_k}{\partial \rho} \right] dV(x). \end{aligned}$$

By applying Fatou's lemma, having in mind (3.4) and using (3.1), letting  $k \rightarrow \infty$ , we obtain

$$\begin{aligned} (3.6) \quad \limsup_{k \rightarrow \infty} \int_{\alpha \leq |x| \leq s} \left[ \Delta_0 R_k + 2(n-2)(1-\rho^2)^{-1} \rho \frac{\partial R_k}{\partial \rho} \right] dV(x) \\ \geq \int_{\alpha \leq |x| \leq s} \left[ \Delta_0 R + 2(n-2)(1-\rho^2)^{-1} \rho \frac{\partial R}{\partial \rho} \right] dV(x) \\ = \int_{S^{n-1}} \int_a^s \rho^{n-1} \frac{\sinh(2R)}{2} \|\nabla \Theta\|_2^2 d\rho d\mathcal{H}^{n-1}. \end{aligned}$$

From (3.5) and (3.6) we obtain

$$\begin{aligned} (3.7) \quad \int_{|\zeta|=s} \frac{\partial R}{\partial \rho} d\mathcal{H}^{n-1}(\zeta) + 2(n-2) \int_{S^{n-1}} \frac{s^n}{1-s^2} (\beta' - \alpha') d\mathcal{H}^{n-1} \\ \geq \int_{S^{n-1}} \int_a^s \rho^{n-1} \frac{\sinh(2R)}{2} \|\nabla \Theta\|_2^2 d\rho d\mathcal{H}^{n-1}. \end{aligned}$$

It follows that

$$\begin{aligned} s^{n-1} \frac{\partial}{\partial s} \int_{|\zeta|=1} R(s\zeta) d\mathcal{H}^{n-1}(\zeta) + \int_{S^{n-1}} \frac{2(n-2)s^n}{1-s^2} (\beta' - \alpha') d\mathcal{H}^{n-1} \\ \geq \int_{S^{n-1}} \int_a^s \rho^{n-1} \frac{\sinh(2R)}{2} \|\nabla \Theta\|_2^2 d\rho d\mathcal{H}^{n-1} \end{aligned}$$

i.e.

$$\begin{aligned} s^{n-1} \frac{\partial}{\partial s} \int_{|\zeta|=1} R(s\zeta) d\mathcal{H}^{n-1}(\zeta) + 2(n-2)\omega_{n-1} \frac{s^n}{1-s^2} (\beta' - \alpha') \\ \geq \int_{S^{n-1}} \int_a^s \rho^{n-1} \frac{\sinh(2R)}{2} \|\nabla\Theta\|_2^2 d\rho d\mathcal{H}^{n-1} \end{aligned}$$

or what is the same

$$\begin{aligned} \frac{\partial}{\partial s} \int_{|\zeta|=1} R(s\zeta) dS(\zeta) + 2(n-2)\omega_{n-1} \frac{s}{1-s^2} (\beta' - \alpha') \\ \geq s^{1-n} \int_{S^{n-1}} \int_a^s \rho^{n-1} \frac{\sinh(2R)}{2} \|\nabla\Theta\|_2^2 d\rho d\mathcal{H}^{n-1}. \end{aligned}$$

Integrating the previous expression w.r.t  $s$  on  $[a, b]$  we obtain

$$\begin{aligned} (3.8) \quad \omega_{n-1}(\beta' - \alpha') + (n-2)\omega_{n-1} \log \frac{1-a^2}{1-b^2} (\beta' - \alpha') \\ \geq \int_a^b s^{1-n} \int_{S^{n-1}} \int_a^s \rho^{n-1} \frac{\sinh(2R)}{2} \|\nabla\Theta\|_2^2 d\rho d\mathcal{H}^{n-1} ds \\ \geq \frac{\sinh(2\alpha')}{2} \int_a^b s^{1-n} \int_{S^{n-1}} \int_a^s \rho^{n-1} \|\nabla\Theta\|_2^2 d\rho d\mathcal{H}^{n-1} ds. \end{aligned}$$

Now we put  $n = 3$ , which implies this simple fact  $n - 1 = 2$ . Combining now (3.8) with Proposition 2.2 we obtain

$$(\beta' - \alpha') + (n-2) \log \frac{1-a^2}{1-b^2} (\beta' - \alpha') \geq \sinh(2\alpha') \int_a^b s^{-2} (s-a) ds$$

and therefore

$$(3.9) \quad (\beta' - \alpha') \left( 1 + \log \frac{1-a^2}{1-b^2} \right) \geq \sinh(2\alpha') \left( -1 + \frac{a}{b} + \log \frac{b}{a} \right).$$

By using the formulas  $a' = 2 \tanh^{-1} a$  and  $b' = 2 \tanh^{-1} b$ , we obtain (1.7), for  $x_0 = 0$  and  $y_0 = 0$ . The general case follows from Proposition 1.2. By using the following formulas

$$2 \tanh^{-1} t = \log \frac{1+t}{1-t} \text{ and } \sinh(4 \tanh^{-1} t) = \frac{4t(1+t^2)}{(1-t^2)^2},$$

we obtain (1.3). This finishes the required proofs.  $\square$

**Example 3.2.** Assume that

$$u(x) = r(\rho) \frac{x}{|x|}$$

is a hyperbolic harmonic mapping. Then from (3.1), taking  $\rho = e^t$ , we obtain that  $r(\rho) = \tanh \frac{y(t)}{2}$  where  $y$  is a solution of the differential equation

$$y'' + (2-n) \coth(t) y' = \frac{(n-1) \sinh(2y)}{2}.$$

Two of many possible solutions of the previous differential equation are  $y_+(t) = 2 \tanh^{-1}(e^t)$  and  $y_-(t) = 2 \tanh^{-1}(e^{-t})$ . The function  $y_+$  produces the identity mapping  $u_+(x) = x$ , while the function  $y_-$  produces the inversion  $u_-(x) = x/|x|^2$ .

Both are hyperbolic harmonic mappings, but the second one maps the complement of the unit ball onto the unit ball. Notice that both, the unit ball and its complement, with appropriate metrics, can identify the hyperbolic space  $\mathbf{H}^n$ .

## REFERENCES

- [1] AHLFORS, L. *Möbius transformations in several dimensions*. Ordway Professorship Lectures in Mathematics. University of Minnesota, School of Mathematics, Minneapolis, Minn., 1981. ii+150 pp.
- [2] ASTALA, K.; IWANIEC, T.; MARTIN, G.: *Deformations of annuli with smallest mean distortion*. Arch. Ration. Mech. Anal. **195** (2010), no. 3, 899–921.
- [3] HAN, Z.-C.: *Remarks on the geometric behavior of harmonic maps between surfaces*. Chow, Ben (ed.) et al., *Elliptic and parabolic methods in geometry*. Proceedings of a workshop, Minneapolis, MN, USA, May 23–27, 1994. Wellesley, MA: A K Peters. 57–66 (1996).
- [4] IWANIEC, T.; ONNINEN, J. *n-Harmonic Mappings Between Annuli: The Art of Integrating Free Lagrangians*, to appear in Mem. Amer. Math. Soc.
- [5] IWANIEC, T., KOVALEV, L. K., ONNINEN, J.: *The Nitsche conjecture*, J. Amer. Math. Soc. **24** (2011), no. 2, 345–373.
- [6] KALAJ, D.: *On the univalent solution of PDE  $\Delta u = f$  between spherical annuli*. J. Math. Anal. Appl. **327** (2007), no. 1, 1–11.
- [7] KALAJ, D.: *On the Nitsche conjecture for harmonic mappings in  $\mathbb{R}^2$  and  $\mathbb{R}^3$* . Israel J. Math. **150** (2005), 241–251.
- [8] KALAJ, D.: *Deformations of annuli on Riemann surfaces with smallest mean distortion*. arXiv:1005.5269.
- [9] KALAJ, D.: *Harmonic maps between annuli on Riemann surfaces*. Israel J. Math. **182**, (2011), 123–147. (arXiv:1003.2744).
- [10] MARTIN, G., MCKUBRE-JORDENS, M. *Deformations with smallest weighted  $L_p$  average distortion and Nitsche type phenomena*. Journal of the London Mathematical Society, (in press).
- [11] LI, P., TAM, L.-F.: *The heat equation and harmonic maps of complete manifolds*. Invent. Math. **105**, No.1, 1–46 (1991).
- [12] LI, P., TAM, L.-F.: *Uniqueness and regularity of proper harmonic maps*. Ann. Math. (2) **137**, No.1, 167–201 (1993).
- [13] LYZZAIK, A.: *The modulus of the image of annuli under univalent harmonic mappings and a conjecture of J. C. C. Nitsche*. J. London Math. soc., (2) **64** (2001), pp. 369–384.
- [14] NITSCHKE, J.C.C.: *On the modulus of doubly connected regions under harmonic mappings*, Amer. Math. Monthly **69** (1962), 781–782.
- [15] SCHOEN, R.; YAU, S. T.: *Lectures on harmonic maps*. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997. vi+394 pp.
- [16] VUORINEN, M.: *Conformal geometry and quasiregular mappings*. Lecture Notes in Mathematics, 1319. Springer-Verlag, Berlin, 1988. xx+209 pp.
- [17] WEITSMAN, A.: *Univalent harmonic mappings of annuli and a conjecture of J.C.C. Nitsche*. Israel J. Math. **124**, 327–331 (2001)

FACULTY OF NATURAL SCIENCES AND MATHEMATICS, UNIVERSITY OF MONTENEGRO, CETINJSKI  
PUT B.B. 81000 PODGORICA, MONTENEGRO  
E-mail address: davidk@t-com.me